Garside groups and Translation numbers

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(joint work with Eon-Kyung Lee)

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A survey of three papers.

- S. J. Lee, *Garside groups are strongly translation discrete*, J. Algebra, 309 (2007), 594-609.
- E.-K. Lee and S. J. Lee, *Translation numbers in a Garside group are rational with uniformly bounded denominators*, arXiv:math.GT/0604061, to appear in J. Pure Appl. Algebra.
- E.-K. Lee and S. J. Lee, *Some power of an element in a Garside group is conjugate to a periodically geodesic element*, arXiv:math.GN/0604144, to appear in Bull. LMS.

Plan

1. Translation numbers in groups
2. Our results on the translation discreteness of Garside groups
3. Brief introduction to Garside groups
4. Sketch of proofs

Translation numbers in combinatorial groups

Introduced by Gersten and Short in 1991.

G: a group
X: a finite set of generators for G.
$|\cdot|_X$: shortest word length in the alphabet $X \cup X^{-1}$.

The translation number of $g \in G$ with respect to X is

$$t_X(g) = \lim_{n \to \infty} \frac{|g^n|_X}{n}.$$

Basic properties of $t_X(g) = \lim_{n\to\infty} \frac{|g^n|_X}{n}$

- $0 \leqslant t_X(g) \leqslant |g|_X$ for all $g \in G$.

  $0 \leqslant |g^n|_X \leqslant n|g|_X$ implies $0 \leqslant t_X(g) \leqslant |g|_X$.

- Translation number is a conjugacy invariant.

  $$(h^{-1}gh)^n = h^{-1}g^n h$$
  $$\Rightarrow |g^n|_X - 2|h|_X \leqslant |(h^{-1}gh)^n|_X \leqslant |g^n|_X + 2|h|_X$$
  $$\Rightarrow t_X(h^{-1}gh) = t_X(g)$$

- Translation number is a semi-norm on abelian subgroups.

  – $t_X(g^n) = |n| \cdot t_X(g)$ for $n \in \mathbb{Z}$.
  – If $gh = hg$, then $|(gh)^n|_X \leqslant |g^n h^n|_X \leqslant |g^n|_X + |h^n|_X$.
    Therefore, $t_X(gh) \leqslant t_X(g) + t_X(h)$.

Rmk. If $t_X(g) > 0$ for $g \neq 1$, then $t_X(\cdot)$ is a norm on abelian subgroups.

Examples

- G: a free abelian group $\langle x_1, \ldots, x_n \mid x_i x_j = x_j x_i \rangle$; $X = \{x_1, \ldots, x_n\}$.

  If $g = x_1^{k_1} \cdots x_n^{k_n}$, then $t_X(g) = |g|_X = |k_1| + \cdots + |k_n|$, the l_1-norm of g.

- G: a free group, freely generated by X.

  Then, $t_X(g)$ is the length of a cyclically reduced word representing g.

- M: a closed Riemannian manifold of non-positive curvature.

  $g \in \pi_1(M)$ acts on the universal cover $\tilde{M}$ as an isometry.
  There is a bi-infinite geodesic A_g in $\tilde{M}$ invariant under the action of g.
  The action of g on A_g is by translation by a positive real $\ell(g)$.
  There exists a positive constant $\lambda = \lambda(M, X)$ such that

$$\frac{1}{\lambda}\, t_X(g) \leqslant \ell(g) \leqslant \lambda\, t_X(g), \quad \text{for all } g \in \pi_1(M).$$

Discreteness properties of translation numbers

A finitely generated group G is said to be

- translation separable if $t_X(g) > 0$ for non-torsion elements g;
- translation discrete if there is a constant $\epsilon = \epsilon(G, X) > 0$ such that $t_X(g) > \epsilon$ for any non-torsion element g;
- strongly translation discrete if G is translation separable and for any $r > 0$, there are only finitely many conjugacy classes with $t_X(g) < r$.

Translation discreteness implies some good properties

- (Gersten-Short, '91)
  G: translation separable
  $\Rightarrow$ Every solvable subgroup of G is f.g. and virtually abelian.

- (I. Kapovich '97)
  G: translation discrete
  $\Rightarrow$ G cannot contain subgroups isomorphic to $\mathbb{Q}$ or the group of p-adic numbers $\mathbb{Q}_p = \{k/p^l \mid k \in \mathbb{Z},\ l \in \mathbb{N}\}$.

- (Conner '00)
  G: translation separable, solvable, of finite vcd
  $\Rightarrow$ G is metabelian by finite.

  G: translation discrete
  $\Rightarrow$ Every solvable subgroup of finite vcd is a finite extension of $\mathbb{Z}^m$.

Translation discreteness of geometric groups

- (Gersten-Short '91)
  Biautomatic groups are translation separable.

- (I. Kapovich '97)
  C(4)-T(4)-P, C(3)-T(6)-P and C(6)-P small cancellation groups are strongly translation discrete.
  (These groups are biautomatic but not necessarily word hyperbolic.)

- (E. Alibegović '00)
  $\mathrm{Out}(F_n)$ is translation discrete.

Translation discreteness of hyperbolic groups

- (Gromov '87, Swenson '95)
  Hyperbolic groups are strongly translation discrete. Moreover,
  (i) translation numbers are rational with uniformly bounded denominators;
  (ii) for every element g, some power g^m is conjugate to a periodically geodesic element.

Definition. g is periodically geodesic if $|g^n|_X = n\,|g|_X$ for all $n \geqslant 1$.
(Note that if g is periodically geodesic, then $t_X(g) = |g|_X$, hence it is an integer.)

Translation discreteness of Garside groups

- (Bestvina '99)
  Artin groups of finite type are translation discrete.

- (Charney, Meier and Whittlesey '04)
  Garside groups *with a tame Garside element* are translation discrete.

  Garside groups are endowed with a special element, called a Garside element. The tameness of a Garside element is a mild condition. It is conjectured that every Garside group has a tame Garside element.

Our results on translation discreteness of Garside groups

Translation discreteness of Garside groups is as good as hyperbolic groups

- (L, J. Algebra 2007)
  All Garside groups are strongly translation discrete.

- (E.Lee-L, preprints 2006)
  (i) Translation numbers of elements in a Garside group are rational with uniformly bounded denominators.

  (ii) For every element g of a Garside group, some power g^m is conjugate to a periodically geodesic element.

Here, the generating set X is required to be the *set of simple elements* which arises from the Garside structure.

Applications

Let G be a Garside group.

- Every solvable subgroups of G are f.g. and virtually abelian.

  ($\because$ translation separable)

- G cannot contain subgroups isomorphic to $\mathbb{Q}$ or the group of p-adic numbers $\mathbb{Q}_p = \{k/p^l \mid k \in \mathbb{Z},\ l \in \mathbb{N}\}$. ($\because$ translation discrete)

- The following problems are solvable. ($\because$ strongly translation discrete)
  - Root problem: given $g \in G$ and $n \geqslant 1$, find $h \in G$ s.t. $h^n = g$.
  - Power problem: given $g, h \in G$, find $n \in \mathbb{Z}$ s.t. $h^n = g$.
  - Proper power problem: given $g \in G$, find $h \in G$ and $n \geqslant 2$ s.t. $h^n = g$.

- An easy proof of Algebraic Flat Torus Theorem for Garside groups: every abelian subgroup of a Garside group is quasi-isometric to $\mathbb{Z}^n$.

  ($\because$ translation separable & ultra summit property of abelian subgroups)

- Applications to the conjugacy problem and the reducibility problem in braid groups. ($\because$ properties of another invariant $t_{\inf}(\cdot)$)

Conjugacy problem: given two elements g and h, decide whether they are conjugate, and, if so, find a conjugating element x such that $h = x^{-1}gx$.

An n-braid is said to be reducible if it leaves invariant an essential curve system in the n-punctured disk D_n.

Reducibility problem: given an n-braid, decide whether it is reducible, and if so, find a reduction system.

Brief introduction to Garside groups

Garside groups

- a lattice-theoretic generalization of braid groups and Artin groups of finite typ introduced by Dehornoy and Paris in 1999.
- torsion-free, bi-automatic, solvable word/conjugacy problem, finite $K(\pi, 1)$.
- We use the following notations.
  * G: a Garside group
  * G^+: positive monoid
  * Δ: Garside element
  * $\mathcal{D}$: set of simple elements
  * $\leqslant_L, \leqslant_R$: partial orderings
  * $\wedge_L, \vee_L, \wedge_R, \vee_R$: lattice operations.

Definition of Garside monoids and groups

A finitely generated monoid M is called a Garside monoid if

- M is atomic;
- M is cancellative;
- the posets $(M, \leqslant_L)$ and $(M, \leqslant_R)$ are lattices;
- there exists a Garside element Δ.

A Garside group is defined as the group of fractions of a Garside monoid.

Atomicity

An element $a \neq 1$ of a monoid M is called an atom (or an indecomposible element) if $a = bc$ implies either $b = 1$ or $c = 1$.

$\|a\| = \sup\{$lengths of all expressions of a in terms of atoms$\}$.

If $M = \langle x, y, z \mid x^2 = y^3 = z^5 \rangle^+$, then $\|x^2\| = \|z^5\| = 5$.
If $M = \langle x \mid x^n = 1 \rangle^+$, then $\|1\| = \|x^n\| = \|x^{2n}\| = \cdots = \infty$.

A monoid M is atomic if

M is generated by its atoms;
$\|a\| < \infty$ for all $a \in M$.

Cancellative

A monoid M is cancellative if $aba' = aca'$ implies $b = c$.

e.g. $\langle a, b, c \mid ab = ac \rangle^{+}$ is not cancellative because $ab = ac$ but $b \neq c$.

Note that if a monoid M embeds in a group, then M is cancellative.

Partial orderings and lattice structures

Recall that for a monoid M to be a Garside monoid, the posets $(M, \leqslant_L)$ and $(M, \leqslant_R)$ must be lattices.

For $a, b \in M$, define

$a \leqslant_L b$ if $b = ac$ for some $c \in M$. (a is called a prefix of b.)

$a \leqslant_R b$ if $b = ca$ for some $c \in M$. (a is called a suffix of b.)

If M is atomic, then $(M, \leqslant_L)$ and $(M, \leqslant_R)$ are posets.

If $a \leqslant_L b$ and $b \leqslant_L c$, then $a \leqslant_L c$: it is obvious.
If $a \leqslant_L b$ and $b \leqslant_L a$, then $a = b$: it is a consequence of atomicity.

Since $a = bd_1$ and $b = ad_2$ for some $d_1, d_2 \in M$,

$$a = a(d_2 d_1) = a(d_2 d_1)^2 = \cdots .$$

Since $\|a\| < \infty$, we have $d_1 = d_2 = 1$, hence $a = b$.

Recall that a poset $(P, \leqslant)$ is a lattice if there exist gcd and lcm.

The gcd of $a, b \in P$ is the unique element $a \wedge b$ such that

$a \wedge b \leqslant a$; $a \wedge b \leqslant b$; if $c \leqslant a$ and $c \leqslant b$ then $c \leqslant a \wedge b$.

The lcm $a \vee b$ is similarly defined.

Remark.

- The defining relations in the Artin presentation of braid groups show the lcm of each pair of generators.

  $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ can be interpreted as $\sigma_1 \vee_L \sigma_2 = \sigma_1\sigma_2\sigma_1$.

- Bestvina interpreted these lattice structures as an NPC (non-positively curved aspect) of Artin groups of finite type. His work was generalized to Garside groups by Charney, Meier and Whittlesey.

Garside element Δ

An element Δ is called a Garside element if

(i) for each $a \in M$, $a \leqslant_L \Delta$ if and only if $a \leqslant_R \Delta$;
(ii) the set $\mathcal{D} = \{a \in M : a \leqslant_L \Delta\}$ generates M.

Elements of $\mathcal{D}$ are called simple elements.

Recall that a finitely generated monoid M is a Garside monoid if

– M is atomic;
– M is cancellative;
– the posets $(M, \leqslant_L)$ and $(M, \leqslant_R)$ are lattices;
– there exists a Garside element Δ,

and that a Garside group is defined as the group of fractions of a Garside monoid.

Group of fractions

A Garside group is defined as the group of fractions of a Garside monoid.

- Garside monoids satisfy Ore condition (cancellative and having a right multiple of any two elements), hence they embed in their group of fractions.
- Group of fractions is similar to the field of quotients of commutative rings.
  - Give an equivalence relation on $M \times M$ by
    $(a, b) \sim (c, d)$ if $au = cv$ and $bu = dv$ for some $u, v \in M$.
  - Let ab^{-1} be the equivalence class of (a, b).
    (Note that $ab^{-1} = \frac{au}{bu} = \frac{cv}{dv} = cd^{-1}$.)
  - Define $(ab^{-1}) \cdot (cd^{-1}) = (au)(dv)^{-1}$, where $bu = cv$.
    (Note that $ab^{-1}cd^{-1} = \frac{a}{b} \cdot \frac{c}{d} = \frac{au}{bu} \cdot \frac{cv}{dv} = \frac{au}{dv} = (au)(dv)^{-1}$.)
  - Then $G = \{ab^{-1} : a, b \in M\}$ is a group and M embeds in G.

Examples of Garside groups

- Braid group $B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{l} \sigma_i\sigma_j = \sigma_j\sigma_i, \ (|i-j| \geqslant 2) \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \end{array} \right\rangle$.

  $B_n^+ =$ monoid of positive words in σ_i's modulo braid relations.
  $\Delta = (\sigma_{n-1} \cdots \sigma_1) \cdots (\sigma_2\sigma_1)\sigma_1$.

- Artin groups of finite type

$$A = \langle x_1, \ldots, x_n \mid \underbrace{x_i x_j x_i \cdots}_{m_{ij}} = \underbrace{x_j x_i x_j \cdots}_{m_{ij}} \rangle, \qquad A/\langle x_i^2 \rangle \text{ is a finite group.}$$

- Some groups such as $\langle x_1, \ldots, x_n \mid x_1^{p_1} = \cdots = x_n^{p_n} \rangle$, $p_i \geqslant 2$.

- (Picantin '01, L '07)
  Semidirect products (crossed products) of Garside monoids are Garside monoids.

Infimum and supremum

For $g \in G$, there are integers r and s such that $\Delta^r \leqslant_L g \leqslant_L \Delta^s$.
The invariants $\inf(g)$ and $\sup(g)$ are defined by

$$\inf(g) = \max\{r \in \mathbb{Z} \mid \Delta^r \leqslant_L g\};$$
$$\sup(g) = \min\{s \in \mathbb{Z} \mid g \leqslant_L \Delta^s\}.$$

Partial orders $\leqslant_L$ and $\leqslant_R$ in the monoid M extends to the group G:

$g \leqslant_L h$ if $ga = h$ for some $a \in M$;
$g \leqslant_R h$ if $ag = h$ for some $a \in M$.

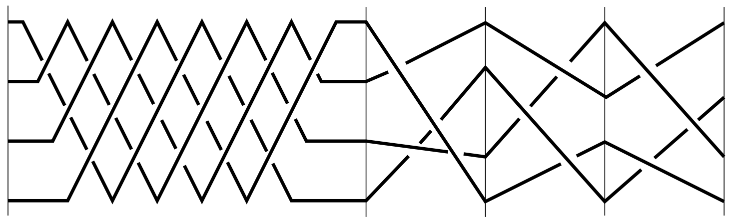
Normal form

Every element $g \in G$ can be uniquely expressed as

$$g = \Delta^u a_1 a_2 \cdots a_k,$$

where $a_i \in \mathcal{D} \setminus \{1, \Delta\}$ and $a_i = \Delta \wedge_L (a_i \cdots a_k)$.

In this case, $\inf(g) = u$ and $\sup(g) = u + k$.



$\Delta^{-3}(\sigma_3\sigma_1\sigma_2\sigma_1)(\sigma_2\sigma_1\sigma_3)(\sigma_1\sigma_3\sigma_2)$

(a 4-braid in normal form)

Conjugacy Problem in Garside groups

Given two elements g and h of a Garside group G,
– decide whether $g \underset{\text{conj}}{\sim} h$ or not;
– if $g \underset{\text{conj}}{\sim} h$, find $x \in G$ such that $h = x^{-1}gx$.

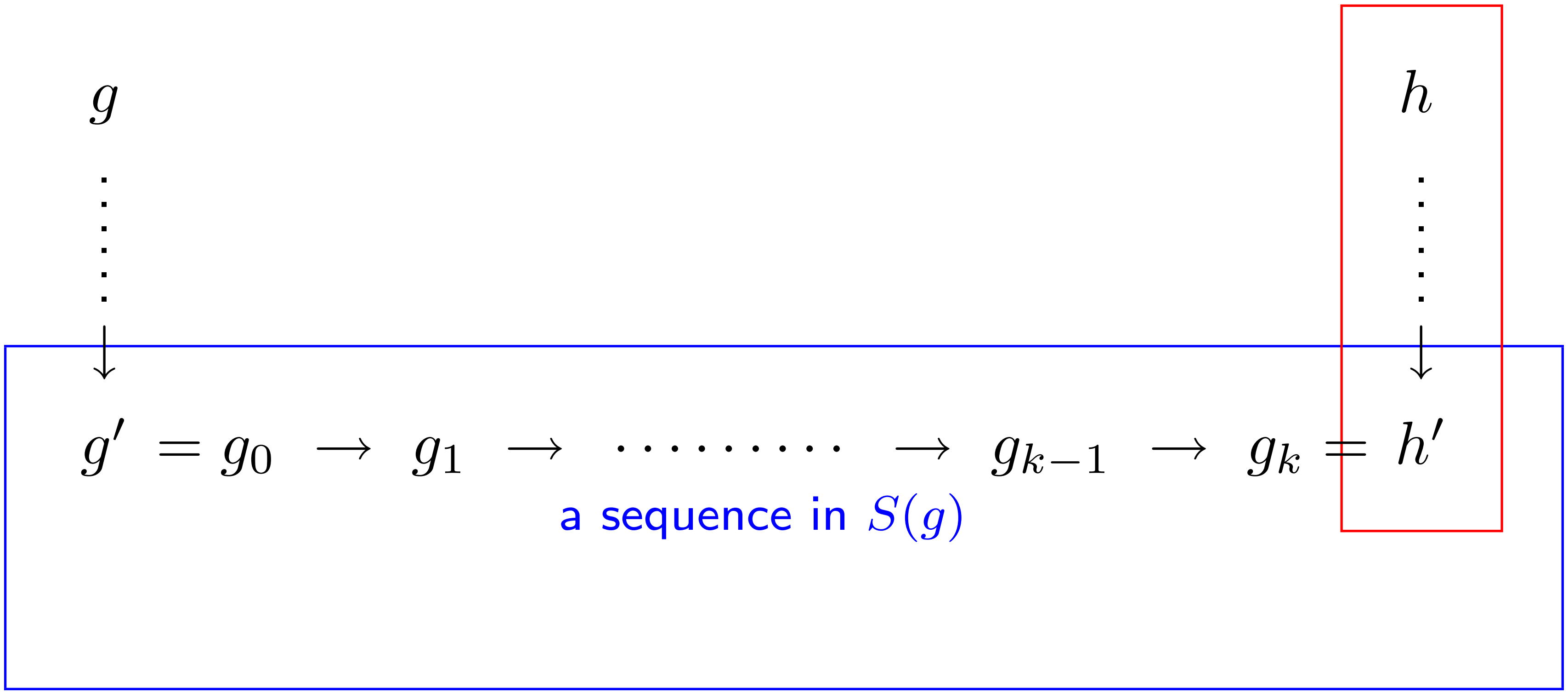
Most solutions to the conjugacy problem in Garside group work as follows.

- Let $S(g)$ be one of the following sets for $g \in G$.

  the summit set [Garside], the super summit set [Thurston, Elrifai-Morton], the ultra summit set [Gebhardt], the reduced super summit set [L, Ko-J.W.Lee], the stable super/ultra summit set [E.Lee-L, BGG].

  Then $S(g)$ is finite and nonempty such that $g \underset{\text{conj}}{\sim} h$ iff $S(g) = S(h)$.

- **Algorithm:** given two elements g and h of G:
  (i) compute $S(g)$;
  (ii) compute an element h' of $S(h)$;
  (iii) check whether $h' \in S(g)$.


g
h
g' = g_0 → g_1 → ········· → g_{k−1} → g_k = h'
a sequence in S(g)

Super summit sets

Let $[g]$ denote the conjugacy class of g. Define

$$\inf_s(g) = \max\{\inf(h) : h \in [g]\};$$
$$\sup_s(g) = \min\{\sup(h) : h \in [g]\}.$$

The super summit set $[g]^S$ is defined as

$$[g]^S = \{h \in [g] : \inf(h) = \inf_s(g),\ \sup(h) = \sup_s(h)\}.$$

Remark.
A super summit element has the shortest normal form in the conjugacy class.

Let $h = \Delta^u a_1 a_2 \cdots a_k$ be conjugate to g.
Then $\sup(h) - \inf(h)$ is the number of simple elements in the normal form of h;
$h \in [g]^S$ iff $\sup(h) - \inf(h)$ is minimal in the conjugacy class.

Cycling and decycling

- Let $g = \Delta^u a_1 \cdots a_k$ be in normal form. (Let $\tau(g) = \Delta^{-1} g \Delta$.)

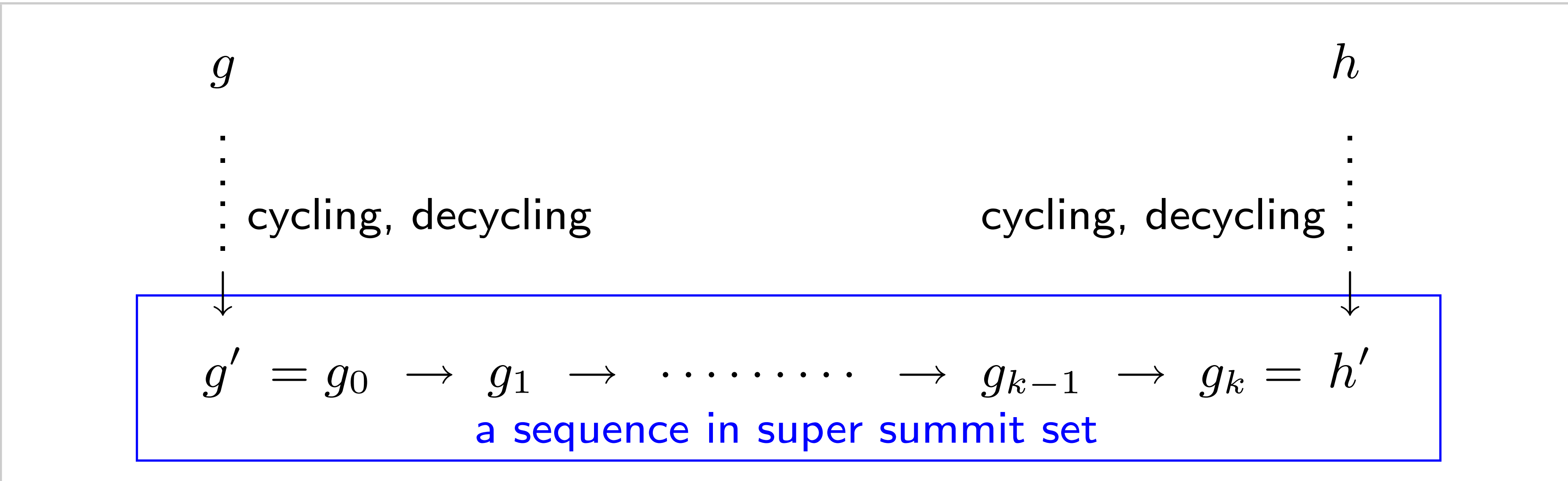
$$\text{cycling, } \mathbf{c}(g) = \Delta^u a_2 \cdots a_k \tau^{-u}(a_1),$$
$$\text{decycling, } \mathbf{d}(a) = \Delta^u \tau^u(a_k) a_1 \cdots a_{k-1}.$$

  RHS is not in normal form, hence it must be rearranged.

- **Cycling Theorem**(Thurston, Elrifai-Morton, Birman-Ko-L)
  A super summit element can be obtained from any element by applying iterated cycling and decycling.

Super summit set algorithm

- **Convexity theorem**(Thurston, Elrifai-Morton, Birman-Ko-L, Franco-González-Meneses, Gebhardt)
  For any elements g' and h' in a super summit set, there is a finite sequence in the super summit set
  $$g' = g_0 \to g_1 \to \cdots \to g_k = h'$$
  such that g_{i+1} is conjugate to g_i by a simple element.



Sketch of proofs of our results on the translation discreteness of Garside groups

Recall our results on translation discreteness of Garside groups

- (L, J. Algebra 2007)
  All Garside groups are strongly translation discrete.

- (E.Lee-L, preprints 2006)
  (i) Translation numbers of elements in a Garside group are rational with uniformly bounded denominators.

  (ii) For every element g of a Garside group, some power g^m is conjugate to a periodically geodesic element.

Here, the generating set X is required to be the set of simple elements which arises from the Garside structure.

Observation: Enough to consider $t_{\inf}(\cdot)$

Define $t_{\inf}(g)$ and $t_{\sup}(g)$ by

$$t_{\inf}(g) = \lim_{n\to\infty} \frac{\inf(g^n)}{n}; \qquad t_{\sup}(g) = \lim_{n\to\infty} \frac{\sup(g^n)}{n}.$$

It is sufficient to consider $t_{\inf}(\cdot)$ to obtain the desired translation discreteness of Garside groups.

- (Charney '95) $|g|_{\mathcal{D}} = \begin{cases} \sup(g) & \text{if } \inf(g) \geqslant 0, \\ -\inf(g) & \text{if } \sup(g) \leqslant 0, \\ \sup(g) - \inf(g) & \text{otherwise.} \end{cases}$
- $t_{\mathcal{D}}(g) = \begin{cases} t_{\sup}(g) & \text{if } \inf_s(g) \geqslant 0, \\ -t_{\inf}(g) & \text{if } \sup_s(g) \leqslant 0, \\ t_{\sup}(g) - t_{\inf}(g) & \text{otherwise.} \end{cases}$
- $\sup(g) = -\inf(g^{-1})$, hence $t_{\sup}(g) = -t_{\inf}(g^{-1})$.

Results on $t_{\inf}(\cdot)$

- (L '07) $\inf_s(g) \leqslant \frac{\inf_s(g^n)}{n} < \inf_s(g) + 1.$

  From the above inequality, it is easy to see the following.
  – $\inf_s(g) \leqslant t_{\inf}(g) \leqslant \inf_s(g) + 1.$
  – $|g|_{\mathcal{D}} - 2 \leqslant t_{\mathcal{D}}(g) \leqslant |g|_{\mathcal{D}}$ for a super summit element g.
  – Garside groups are strongly translation discrete.

- (E.Lee-L '06)

  (i) $t_{\inf}(g)$ is rational of the form p/q with $1 \leqslant q \leqslant \|\Delta\|$.

  (ii) $\inf_s(g) \leqslant t_{\inf}(g) < \inf_s(g) + 1$, hence $\inf_s(g) = \lfloor t_{\inf}(g) \rfloor$.

  Remark. Since $\inf_s(g^n) = \lfloor t_{\inf}(g^n) \rfloor = \lfloor n\, t_{\inf}(g) \rfloor$,
  the sequence $\{\inf_s(g^n)\}_{n \geqslant 1}$ is completely determined by $t_{\inf}(g)$.

Ingredients for proofs

- If M_1 and M_2 are Garside monoids, then $M_1 \ltimes M_2$ is a Garside monoid.
- If $x \in \mathbb{R}$ is irrational, then $\{nx - \lfloor nx \rfloor : n \geqslant 1\}$ is dense in $[0, 1]$.
- Schur's theorem in Ramsey theory(Issai Schur, 1916):
  for each integer $k \geqslant 1$, there exists a number $S(k)$, called *Schur's number*, such that for every partition

  $$\{1, 2, \dots, S(k)\} = T_1 \cup \cdots \cup T_k,$$

  some T_i contains two integers n and m together with $n + m$.

Step I: $\inf_s(g) \leqslant \frac{\inf_s(g^n)}{n} < \inf_s(g) + 1$

- Let $\mathbb{Z} = \langle \delta \rangle$ act on G^n by $(g_1, \ldots, g_n)^\delta = (g_n, g_1, \ldots, g_{n-1})$. Then $G(n) = \mathbb{Z} \ltimes G^n$ is a Garside group with a Garside element $(\delta, (\Delta, \ldots, \Delta))$.

  Basic properties of $G(n)$.

  - $\inf(\delta^k, (g_1, \ldots, g_n)) = \min\{k, \inf(g_1), \ldots, \inf(g_n)\}$.
  - $\inf_s(\delta^k, (g, \ldots, g)) = \inf_s(g)$ if $k \geqslant \inf_s(g)$.
  - If $k \equiv 1 \bmod n$, then

    $(\delta^k, (g_1, \ldots, g_n)) \underset{\text{conj}}{\sim} (\delta^k, (h_1, \ldots, h_n))$ in $G(n)$
    iff $g_1 \cdots g_n \underset{\text{conj}}{\sim} h_1 \cdots h_n$ in G.

Sketch of proof of $\inf_s(g) \leqslant \frac{\inf_s(g^n)}{n} < \inf_s(g)+1$**.**

– Easy to show that $n \inf_s(g) \leqslant \inf_s(g^n)$.
  Let $\inf_s(g) = r$ and suppose $\inf_s(g^n) \geqslant n(r+1)$.

– Let $k \gg 1$ with $k \equiv 1 \bmod n$. Note that $\boxed{\inf_s(\delta^k,(g,\dots,g)) = \inf_s(g) = r.}$

  On the other hand, $\boxed{\inf_s(\delta^k,(g,\dots,g)) \geqslant r+1,}$ because

$$\begin{aligned}
(\delta^k,(g,\dots,g)) &\underset{\text{conj}}{\sim} (\delta^k,(g^n,1,\dots,1)) \\
&\underset{\text{conj}}{\sim} (\delta^k,(a\Delta^{n(r+1)},1,\dots,1)), \text{ for some } a \in G^+ \\
&\underset{\text{conj}}{\sim} (\delta^k,(a\Delta^{r+1},\Delta^{r+1},\dots,\Delta^{r+1})).
\end{aligned}$$

Step II: $t_{\inf}(g)$ is rational with denominators $\leqslant \|\Delta\|$

(i) The fractional part of $t_{\inf}(g)$ cannot be contained in $(0, 1/\|\Delta\|)$.

- (Ko) If $\inf(g^k) = k\inf(g)$ for $1 \leqslant k \leqslant \|\Delta\|$, then $\inf(g^k) = k\inf(g)$ for all $k \geqslant 1$.
- If $\inf_s(g^k) = k\inf_s(g)$ for $1 \leqslant k \leqslant \|\Delta\|$, then $\inf_s(g^k) = k\inf_s(g)$ for all $k \geqslant 1$.

  Use the nonemptiness of the stable super summit set
  $[g]^{St} = \{h \in [g] : h^k \in [g^k]^S$ for all $k \geqslant 1\}$.
  It is proved by Lee-L and Birman-Gebhardt-González-Meneses.

- If $t_{\inf}(g) - \lfloor t_{\inf}(g) \rfloor < \|\Delta\|$, then $\inf_s(g^k) = k\inf_s(g)$ for $1 \leqslant k \leqslant \|\Delta\|$, hence $t_{\inf}(g) = \inf_s(g)$ is an integer.

(ii) $t_{\inf}(g)$ is rational.

If $t_{\inf}(g)$ is irrational, $\{n\, t_{\inf}(g) - \lfloor n\, t_{\inf}(g) \rfloor : n \geqslant 1\}$ is dense in $[0, 1]$.
Hence fractional part of $n\, t_{\inf}(g) = t_{\inf}(g^n)$ belongs to $(0, 1/\|\Delta\|)$ for some n.

Step III: $\inf_s(g) \leqslant t_{\inf}(g) < \inf_s(g) + 1$

"$\inf_s(g) \leqslant \frac{\inf_s(g^n)}{n} < \inf_s(g) + 1$" implies "$\inf_s(g) \leqslant t_{\inf}(g) \leqslant \inf_s(g) + 1$".
Therefore, it is enough to show that $t_{\inf}(g) \neq \inf_s(g) + 1$.

Suppose $t_{\inf}(g) = \inf_s(g) + 1$.
Then $\inf_s(g^{n+m}) = \inf_s(g^n) + \inf_s(g^m) + 1$ for all $n, m \geqslant 1$.

- We may assume that g is a stable super summit element.
  Let $g^n = s_n a_n \Delta^{r_n}$, where $r_n = \inf(g^n)$ and $s_n = \Delta \wedge_L (s_n a_n)$.
  Then $\boxed{r_{n+m} = r_n + r_m + 1}$ for all $n, m \geqslant 1$.

- There exists n, m such that $s_n = s_m = s_{n+m}$.

  Let $\{s^{(i)} : i = 1, \ldots, |\mathcal{D}|\}$ be the set of simple elements.
  Define $T_i = \{n \in \mathbb{N} \mid s_n = s^{(i)}\}$ for $i = 1, \ldots, |\mathcal{D}|$. Then,

$$\mathbb{N} = T_1 \cup T_2 \cup \cdots \cup T_{|\mathcal{D}|}.$$

  By Schur's theorem, there exists T_i that contains
  n, m and $n+m$ for some $n, m \geqslant 1$.

- If $s_n = s_m = s_{n+m}$, then $\boxed{r_{n+m} \geqslant r_n + r_m + 2,}$ a contradiction.

  Note that $g^{n+m} = (s_n a_n \Delta^{r_n})(s_m a_m \Delta^{r_m}) \underset{\text{conj}}{\sim} (a_n \Delta^{r_n} s_m)(a_m \Delta^{r_n} s_n)$.
  $s_n = s_{n+m}$ implies $\inf(a_n \Delta^{r_n} s_m) \geqslant r_n + 1$.
  $s_m = s_{n+m}$ implies $\inf(a_m \Delta^{r_m} s_n) \geqslant r_m + 1$.
  Therefore, $\inf(g^{n+m}) \geqslant (r_n + 1) + (r_m + 1) = r_n + r_m + 2$.

Step IV: Existence of periodically geodesic power up to conjugacy

- If $t_{\inf}(g)$ is an integer, then $\underline{\inf_s(g^k) = k\inf_s(g)}$ for all $k \geqslant 1$.

$$\because \quad \inf_s(g^k) = \lfloor t_{\inf}(g^k)\rfloor = \lfloor k\, t_{\inf}(g)\rfloor = k\, t_{\inf}(g) = k\inf_s(g).$$

- If both $t_{\inf}(g)$ and $t_{\sup}(g)$ are integers, then any element of $[g]^{St}$ is periodically geodesic.

$$\because \quad h \in [g]^{St} \quad \Rightarrow \quad \inf(h^k) = k\inf(h) \text{ and } \sup(h^k) = k\sup(h) \text{ for all } k.$$
$$\Rightarrow \quad |h^k|_{\mathcal{D}} = k\,|h|_{\mathcal{D}} \text{ for all } k \geqslant 1.$$

- Let $t_{\inf}(g) = p_1/q_1$ and $t_{\sup}(g) = p_2/q_2$. Let $m = \mathrm{lcm}(q_1, q_2)$. Then any element of $[g^m]^{St}$ is periodically geodesic.

Some references and related Works

- S. M. Gersten and H. B. Short, *Rational subgroups of biautomatic groups*, Ann. of Math. (2) **134** (1991), no. 1, 125–158.
- M. Gromov, *Hyperbolic groups*, in Essays in group theory, 75–263, Springer, 1987.
- E. Swenson, *Hyperbolic Elements in Negatively Curved Groups*, Geom. Dedicata **55** (1995), 443–451.
- M. Bestvina, *Non-positively curved aspects of Artin groups of finite type*, Geometry and Topology **3** (1999) 269–302.
- R. Charney, J. Meier and K. Whittlesey, *Bestvina's normal form complex and the homology of Garside groups*, Geom. Dedicata **105** (2004), 171–188.
- E.-K. Lee and S. J. Lee, *Abelian subgroups of Garside groups*, arXiv:math.GT/0609683, to appear in Comm. Algebra.
- J. Birman, V. Gebhardt and J. González-Meneses,
  Conjugacy in Garside groups I: Cyclings, powers, and rigidity, arXiv:math.GT/0605230.
  Conjugacy in Garside groups II: Structure of the ultra summit set, arXiv:math.GT/0606652.
  Conjugacy in Garside Groups III: Periodic braids arXiv:math.GT/0609616.

– THANK YOU –

Hidden References

- S. Humphries, *Split braids*, Proc. Amer. Math. Soc. **113** (1991), no. 1, 21–26.
- R. Fenn, D. Rolfsen, and J. Zhu, *Centralisers in the braid group and singular braid monoid*, Enseign. Math. (2) **42** (1996), no. 1-2, 75–96.
- D. Rolfsen, *Braid subgroup normalisers, commensurators and induced representations*, Invent. Math. **130** (1997), no. 3, 575–587.
- J. Birman, A. Lubotzky and J. McCarthy, *Abelian and solvable subgroups of the mapping class groups*, Duke Math **50** (1983), 1107–1120.
- J. Gonzalez-Meneses and B. Wiest, *On the structure of the centralizer of a braid*, Ann. Sci. Ecole Norm. Sup. (4) **37** (2004), no. 5, 729–757.
- E.-K. Lee and S. J. Lee, *A Garside-theoretic approach to the reducibility problem in braid groups*, arXiv:math.GT/0506188.
- M. Bestvina, *Non-positively curved aspects of Artin groups of finite type*, Geometry and Topology **3** (1999) 269–302.