Reduction systems and ultra summit sets of reducible braids

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(Joint work with Eon-Kyung Lee)

Reference: *A Garside-theoretic approach to the reducibility problem in braid groups*, preprint, arXiv:math.GT/0506188.

BRAIDS
National University of Singapore
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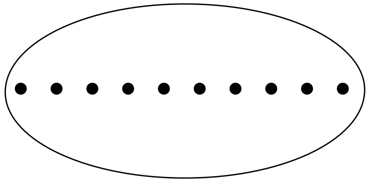
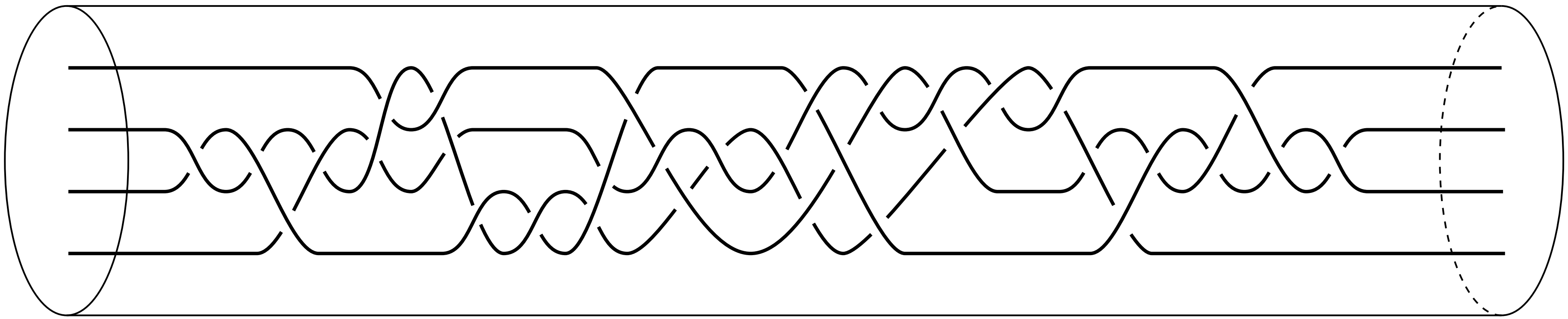
The n-braid group B_n

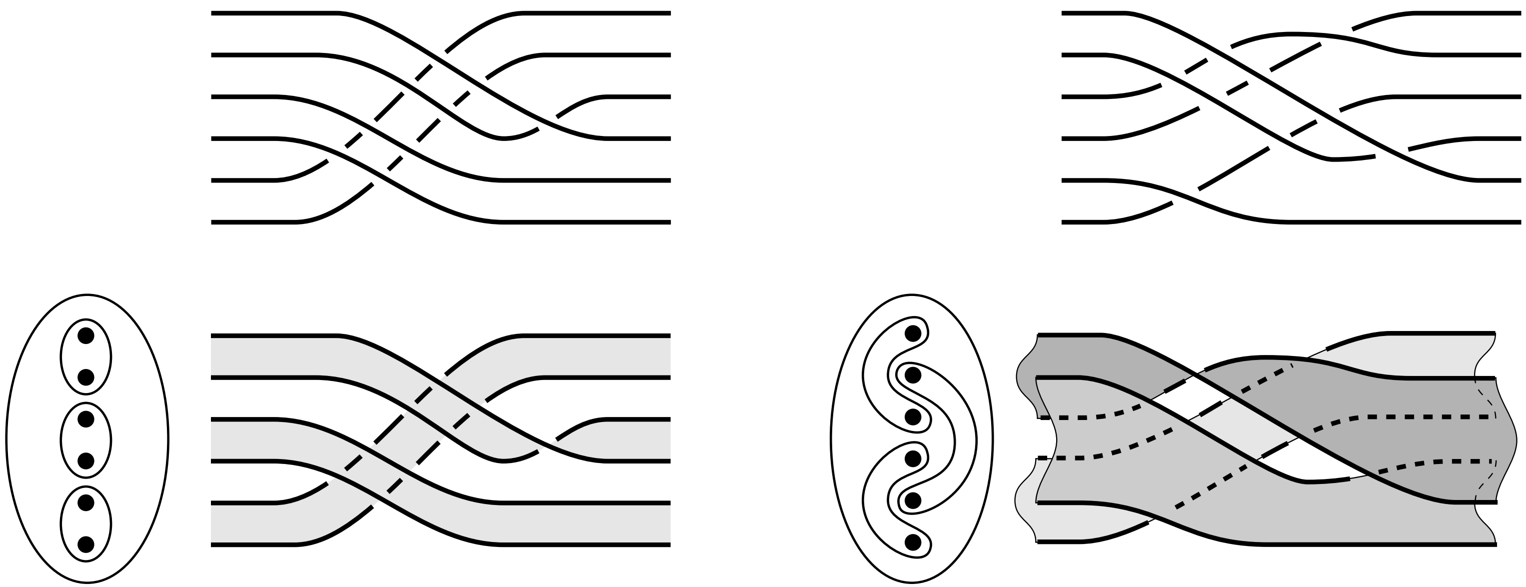
$D_n :=$ n-punctured disc,  .

$B_n :=$ $\mathsf{Homeo}^+(D_n, \partial D^2)/$isotopy rel ∂D^2.



Reducibility problem

$\alpha \in B_n$ is reducible if $\exists$ an essential curve system $\mathcal{C} \subset D_n$ s.t. $\alpha(\mathcal{C}) = \mathcal{C}$.
($\mathcal{C}$ is called a reduction system of α.)



We are interested in the **reducibility problem** in braid groups.

Decision problem: given a braid, decide whether it is reducible.
Search problem: given a reducible braid, find a reduction system.

Motivation

- Nielsen-Thurston Classification Theorem:
  An automorphism of a surface with $\chi < 0$ is, up to isotopy, either *reducible*, *periodic* or *pseudo-Anosov*.

  Geometric and algebraic properties of braid conjugacy classes depends on their dynamical types.

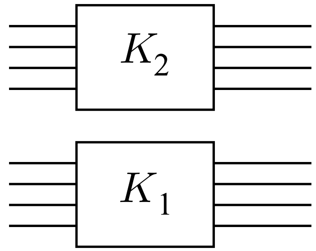
- If we can solve the reducibility problem, then

  we can decide dynamical types of braids;
  we can generalize *some* results on irreducible braids to all braids.

Previous results

- (Humphries '91)

  an algorithm to recognize split braids
  up to a solution to the conjugacy problem.



split braid

- (Bernardete-Nitecki-Gutiérrez '95)

  a complete solution to the reducibility problem
  up to the Garside algorithm to the conjugacy problem.

- (Bestvina-Handel '95)

  the train-track algorithm for surface automorphisms,
  which decides dynamical types and finds geometric structures.

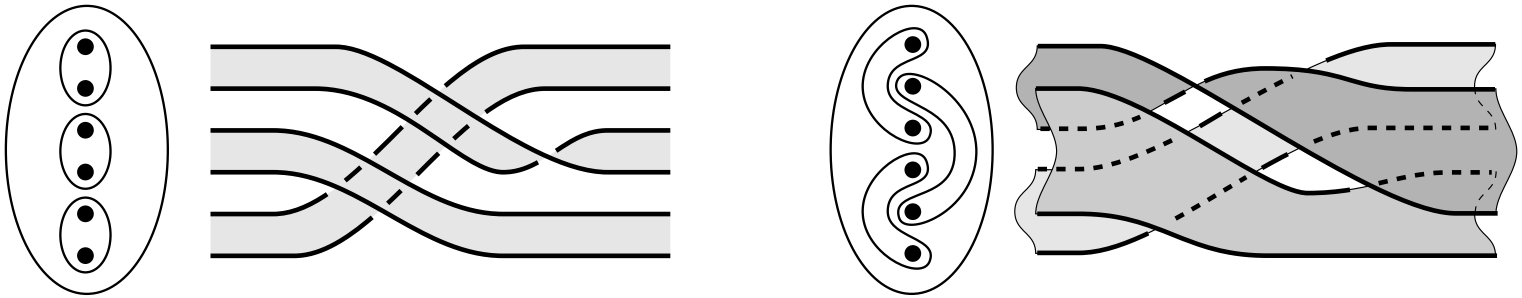
Remark. Above results give algorithms whose computational complexity is exponential with respect to the word length of given braids.

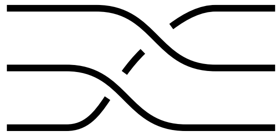
We want a *more efficient* solution.

- we want a polynomial (with respect to both the braid index and the word length of the given braid) time algorithm;
- we give up the train track algorithm and any complete solution to the conjugacy problem in braid groups.

Our result (intuitive)

If $\alpha \in B_n$ is reducible, then we can think of external braid α_{ext}, which is well-defined up to conjugacy.



The external braid of the above braids is .

Theorem (E.-K. Lee, L)

If α_{ext} is simpler than α from a Garside theoretic viewpoint, then we can *easily* find a reduction system of α.

Plan for the talk

1. Garside theory in braid groups
2. Canonical reduction system and standard reduction system
3. Our results

Garside theory in braid groups

The reducibility problem is closely related to the conjugacy problem.

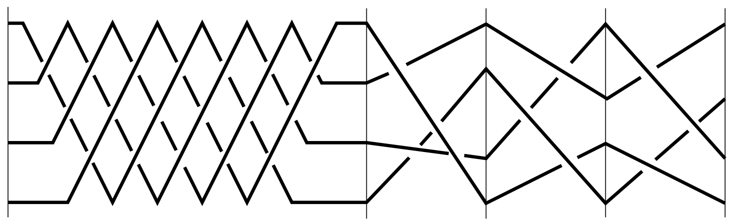
If a curve system $\mathcal{C}$ is invariant under $\alpha \in B_n$,
then $\beta(\mathcal{C})$ is invariant under $\beta\alpha\beta^{-1}$ for any $\beta \in B_n$,
because $\beta\alpha\beta^{-1}(\beta(\mathcal{C})) = \beta\alpha(\mathcal{C}) = \beta(\mathcal{C})$.

Garside (left) normal form

An n-braid α is uniquely expressed as

$$\alpha = \Delta^u A_1 A_2 \cdots A_k,$$

where Δ is the half-twist $\sigma_1(\sigma_2\sigma_1)\cdots(\sigma_{n-1}\cdots\sigma_1)$;
A_i's are permutation braids; $A_i A_{i+1}$ is left-greedy for $i = 1, \ldots, k-1$.



$\Delta^{-3}(\sigma_3\sigma_1\sigma_2\sigma_1)(\sigma_2\sigma_1\sigma_3)(\sigma_1\sigma_3\sigma_2)$

Garside theory to the conjugacy problem

Let $[\alpha]$ denote the conjugacy class of $\alpha \in B_n$.
Let $\alpha = \Delta^u A_1 A_2 \cdots A_k$ be in normal form.

Infimum and supremum.

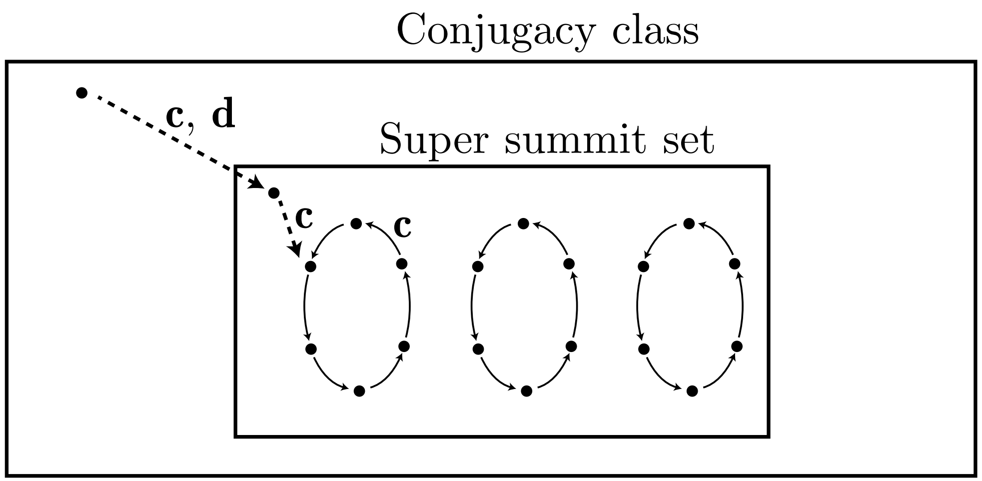
$$\inf(\alpha) = u; \qquad \inf{}_s(\alpha) = \max\{\inf(\beta) : \beta \in [\alpha]\};$$
$$\sup(\alpha) = u + k; \qquad \sup{}_s(\alpha) = \min\{\sup(\beta) : \beta \in [\alpha]\}.$$

Cycling $\mathbf{c}(\alpha)$ and decycling $\mathbf{d}(\alpha)$.

$$\mathbf{c}(\alpha) = \Delta^u A_2 \cdots A_k (\Delta^u A_1 \Delta^{-u});$$
$$\mathbf{d}(\alpha) = \Delta^u (\Delta^{-u} A_k \Delta^u) A_1 \cdots A_{k-1}.$$

Super summit set $[\alpha]^S$ and ultra summit set $[\alpha]^U$.

$$[\alpha]^S = \{\beta \in [\alpha] : \inf(\beta) = \inf_s(\alpha),\ \ \sup(\beta) = \sup_s(\alpha)\};$$
$$[\alpha]^U = \{\beta \in [\alpha]^S : \mathbf{c}^\ell(\beta) = \beta \quad \text{for some } \ell \geqslant 1\}.$$



The ultra summit set is the union of circuits.

- $\beta \in [\alpha]^S$ iff the normal form of β is *shortest* in the conjugacy class.
- **Cycling Theorem**(Thurston, Elrifai-Morton, Birman-Ko-L)
  $\mathbf{c}^k\mathbf{d}^\ell(\alpha) \in [\alpha]^S$ for some $k, \ell \geqslant 1$.

Canonical reduction system

Theorem. (Birman, Lubotzky and McCarthy '83, Ivanov '92)
For reducible braids, there exist **canonical reduction systems**.

Notation.
$\mathcal{R}(\alpha)$: the canonical reduction system of $\alpha \in B_n$;
$\mathcal{R}_{\mathsf{ext}}(\alpha)$: the collection of outermost components of $\mathcal{R}(\alpha)$.

Commuting elements are useful

It is known that

$$\mathcal{R}(\beta\alpha\beta^{-1}) = \beta(\mathcal{R}(\alpha)),$$

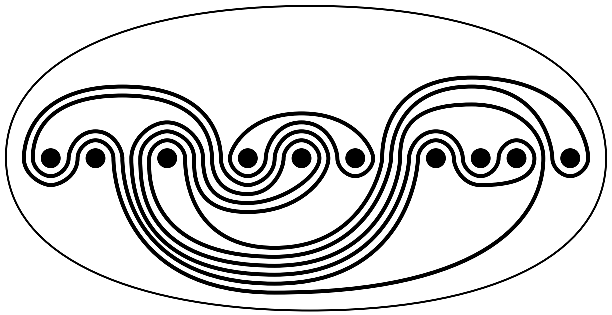
that is, if $\mathcal{C}$ is a CRS of α, then $\beta(\mathcal{C})$ is a CRS of $\beta\alpha\beta^{-1}$.

- If $\alpha\beta = \beta\alpha$, then
  $\mathcal{R}(\alpha)$ is a reduction system of β;
  $\mathcal{R}(\beta)$ is a reduction system of α.

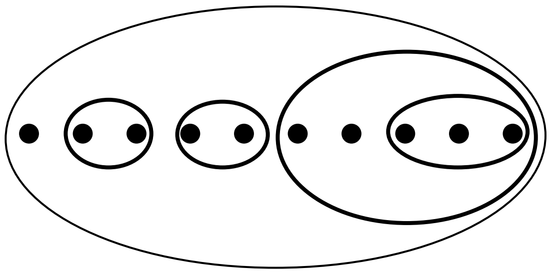
- In order to find a reduction system of α,
  it suffices to find a braid β such that $\alpha\beta = \beta\alpha$,
  together with a component of $\mathcal{R}(\beta)$.

Standard reduction system

A curve system is **standard** if each component is isotopic to a round circle.



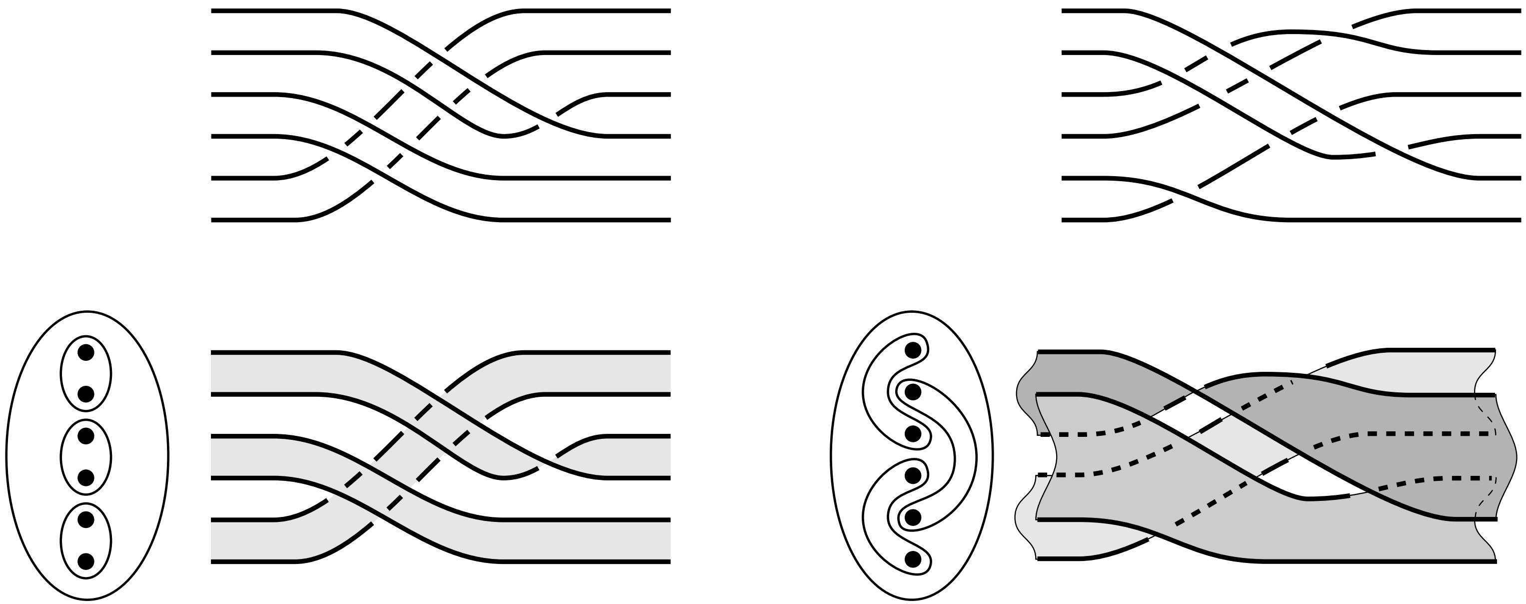
non-standard



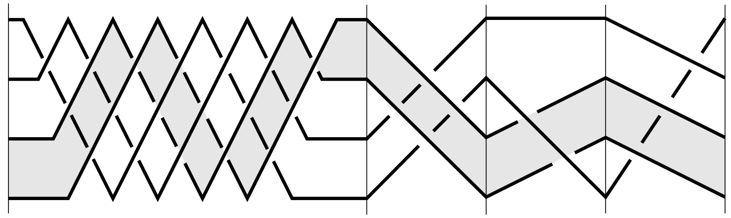
standard

Reducible braids with a standard reduction system

(i) It is easy to find a standard reduction system from braid diagram.



(ii) It is easy to find a standard reduction system from the normal form.



$\Delta^{-3}(\sigma_2\sigma_1\sigma_3\sigma_2)(\sigma_2\sigma_1)(\sigma_1\sigma_2\sigma_3)$

A strategy for solving the reducibility problem:
find a conjugate that has a standard reduction system.

(Bernardete-Nitecki-Gutiérrez '95)
(i) If α has a standard reduction system, then so do $\mathbf{c}(\alpha)$ and $\mathbf{d}(\alpha)$.
(ii) If α is reducible, then there **exists** a braid in $[\alpha]^U$ with a standard reduction system.

Remark. In order to find a reduction system of α using the result of Bernardete-Nitecki-Gutiérrez, we must compute all of $[\alpha]^U$.

Our results

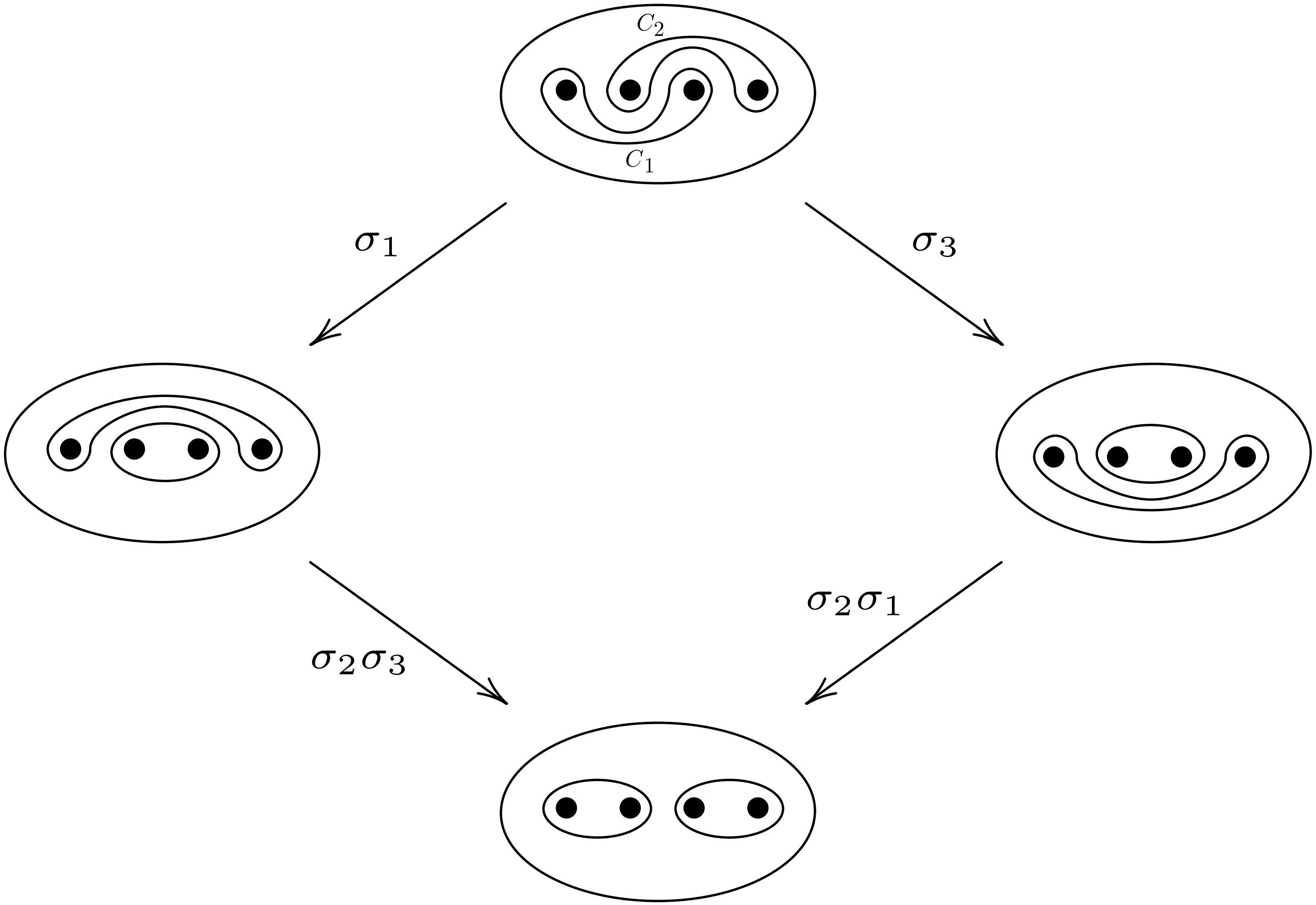
Standardizer

For an essential curve system $\mathcal{C}$, the standardizer of $\mathcal{C}$ is defined as

$$\mathrm{St}(\mathcal{C}) = \{P \in B_n^+ : P(\mathcal{C}) \text{ is standard}\}.$$

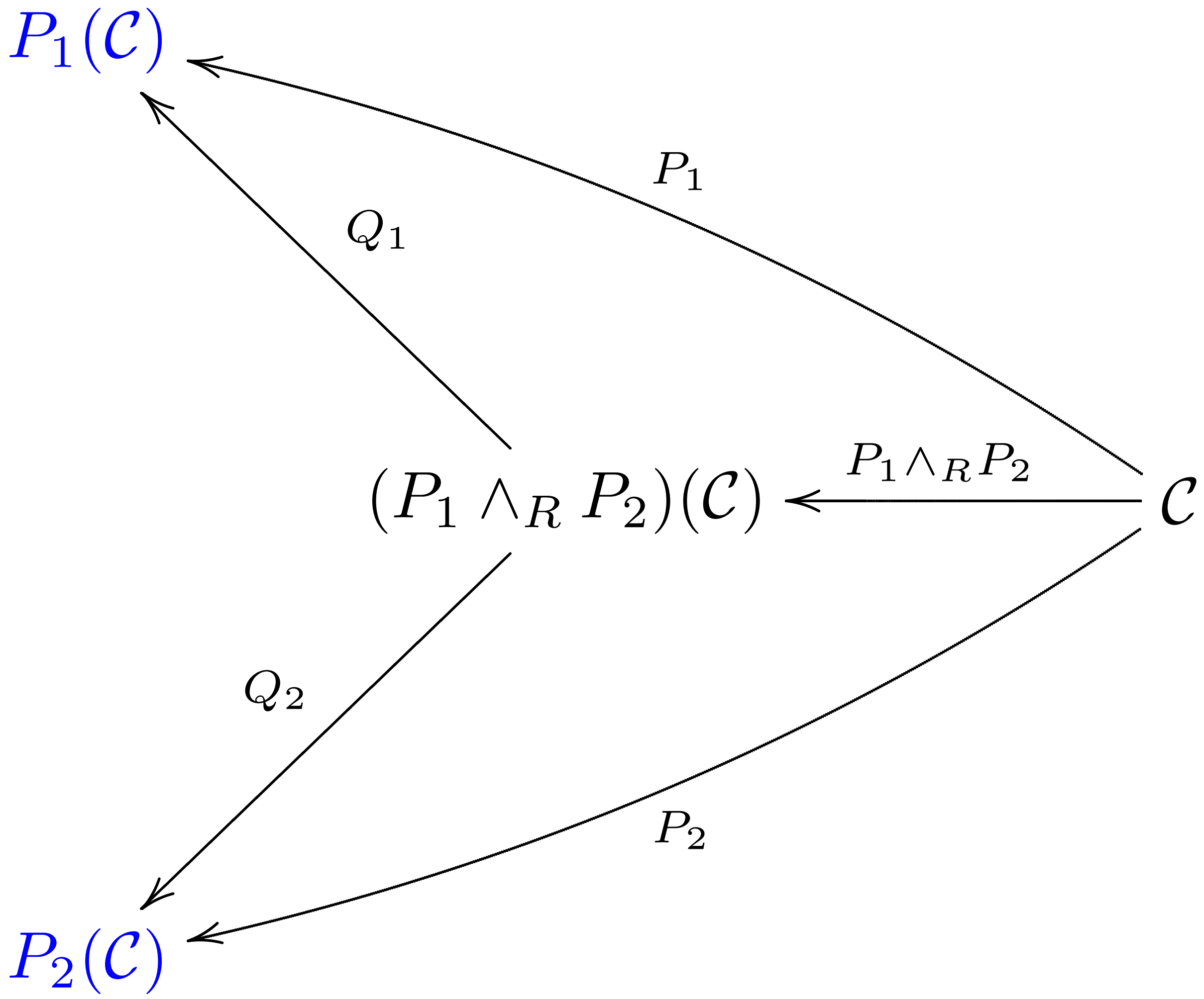
Theorem (E.-K.Lee and L)
$\mathrm{St}(\mathcal{C})$ is closed under $\wedge_R$.

Corollary
In $\mathrm{St}(\mathcal{C})$, there exists a unique positive braid with minimal word length.



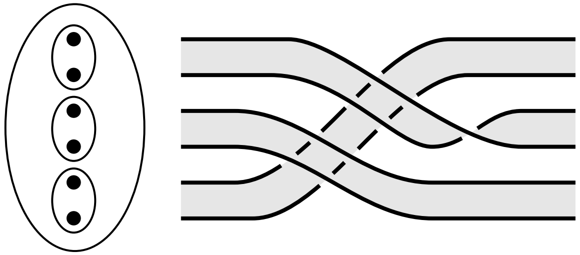
$$\sigma_1 \in \mathsf{St}(C_1), \quad \sigma_3 \in \mathsf{St}(C_2), \quad \sigma_2\sigma_1\sigma_3 \in \mathsf{St}(C_1 \cup C_2)$$

Sketch of Proof. Let $P_1(\mathcal{C})$ and $P_2(\mathcal{C})$ be standard.
Let $P_i = Q_i(P_1 \wedge_R P_2)$ for $i = 1, 2$. Then $P_2(\mathcal{C}) = (Q_2 Q_1^{-1})(P_1(\mathcal{C}))$.

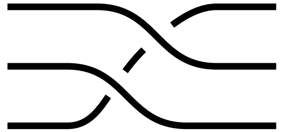


External braids

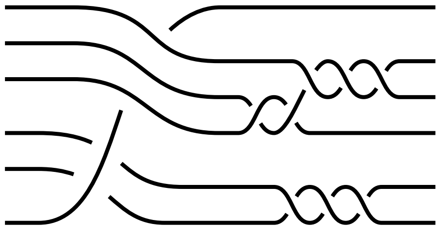
Let $\mathcal{R}_{\mathsf{ext}}(\alpha)$ be standard. The external braid α_{ext} is defined as the restriction of α to the outermost component of $D_n \setminus \mathcal{R}_{\mathsf{ext}}(\alpha)$.



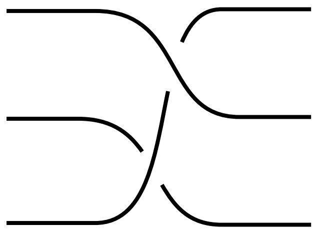
$\alpha = \sigma_2\sigma_1\sigma_3\sigma_2\sigma_4\sigma_3\sigma_5\sigma_4\sigma_3$



$\alpha_{\mathsf{ext}} = \sigma_1\sigma_2$



$\alpha = \sigma_1^{-1}\sigma_2^{-1}\sigma_3\sigma_4\sigma_5\sigma_1^3\sigma_3^2\sigma_4^3\cdots$

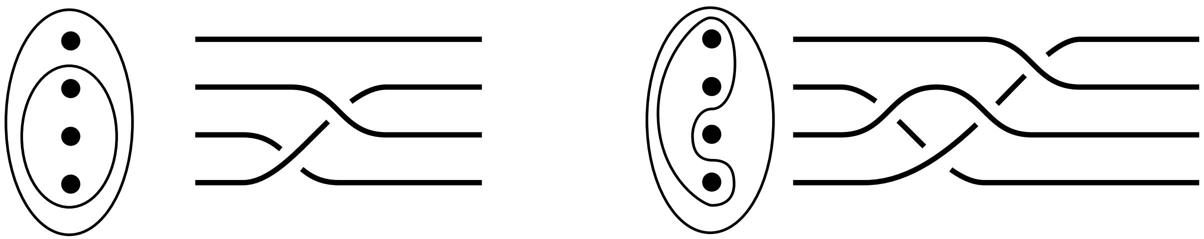


$\alpha_{\mathsf{ext}} = \sigma_1^{-1}\sigma_2\cdots$

If $\mathcal{R}_{\mathsf{ext}}(\alpha)$ is not standard, we first standardize $\mathcal{R}_{\mathsf{ext}}(\alpha)$ using the minimal element of $\mathsf{St}(\mathcal{R}_{\mathsf{ext}}(\alpha))$.

Split braids

Theorem. (E.-K. Lee and L) Let α be a split braid. If the word length of α is minimal in the conjugacy class, then $\mathcal{R}_{\text{ext}}(\alpha)$ is standard.



split braids

Corollary.
(i) If α is a split positive braid, then $\mathcal{R}_{\text{ext}}(\alpha)$ is standard.
(ii) If α commutes with a split positive braid, then α has a standard reduction system.

Main Result

Theorem. (E.-K. Lee and L) If $\inf_s(\alpha_{\mathsf{ext}}) > \inf_s(\alpha)$, then any element $\beta \in [\alpha]^U$ has a standard reduction system.

> $\inf_s(\alpha_{\mathsf{ext}}) > \inf_s(\alpha)$ means that α_{ext} is simpler than α from a Garside-theoretic viewpoint.

Remark. In this case, finding a reduction system is as easy as finding an ultra summit element.

Sketch of proof. By a technical reason, we define $\mathbf{c}_0(\cdot) = \Delta\mathbf{c}(\cdot)\Delta^{-1}$.

Let $\beta \in [\alpha]^U$, hence $\mathbf{c}_0^m(\beta) = \beta$ for some $m \geqslant 1$.
Let $\mathbf{c}_0^{i+1}(\beta) = A_i\mathbf{c}_0^i(\beta)A_i^{-1}$ for a permutation braid A_i.
Let P_i be the minimal element of $\mathrm{St}(\mathcal{R}_{\mathsf{ext}}(\mathbf{c}_0^i(\beta)))$, and let $\gamma_i = P_i\mathbf{c}_0^i(\beta)P_i^{-1}$.

$$
\begin{array}{ccccccccc}
\beta & \xrightarrow{A_0} & \mathbf{c}_0(\beta) & \xrightarrow{A_1} & \mathbf{c}_0^2(\beta) & \xrightarrow{A_2} & \cdots & \xrightarrow{A_{m-1}} & \mathbf{c}_0^m(\beta) = \beta \\
\downarrow{\scriptstyle P_0} & & \downarrow{\scriptstyle P_1} & & \downarrow{\scriptstyle P_2} & & & & \downarrow{\scriptstyle P_m = P_0} \\
\gamma_0 & \xrightarrow{B_0} & \gamma_1 & \xrightarrow{B_1} & \gamma_2 & \xrightarrow{B_2} & \cdots & \xrightarrow{B_{m-1}} & \gamma_m = \gamma_0
\end{array}
$$

$\exists$ permutation braids B_i such that the above diagram commutes.
Let $S = B_{m-1} \cdots B_0$, then S is split positive. $(\because \inf_s(\alpha_{\mathsf{ext}}) > \inf_s(\alpha))$

Let $T = A_{m-1} \cdots A_0$, then T is split positive.
Since $T\beta = \beta T$, $\mathcal{R}_{\mathsf{ext}}(T)$ is a standard reduction system of β.

Corollary.
If α_{ext} is simpler than α from a Garside theoretic viewpoint, then finding a reduction system of α is as easy as finding an element of the ultra summit set of (some power of) α.

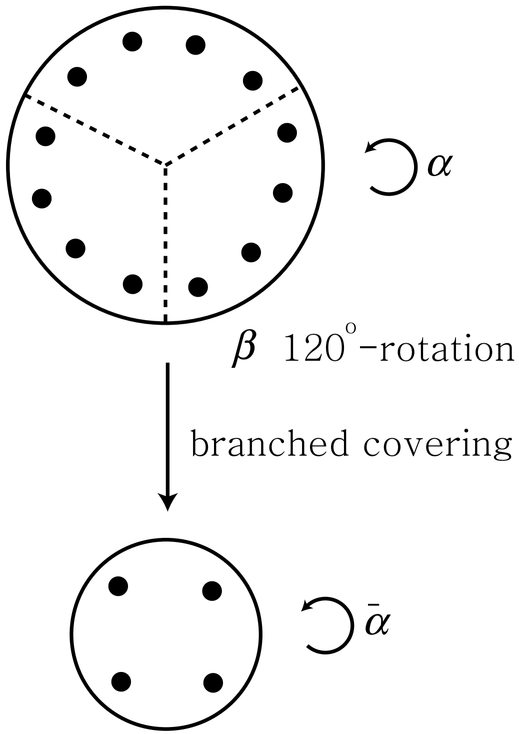
- If $\sup_s(\alpha_{\text{ext}}) < \sup_s(\alpha)$, then each element of $[\alpha]^U_{\mathbf{d}}$ has a standard reduction system.
- If α is a split braid, then each element of $[\alpha]^U \cup [\alpha]^U_{\mathbf{d}}$ has a standard reduction system.
- If α_{ext} is periodic, then there exists $1 \leqslant q < n$ such that each element of $[\alpha^q]^U \cup [\alpha^q]^U_{\mathbf{d}}$ has a standard reduction system.
- If $t_{\inf}(\alpha_{\text{ext}}) > t_{\inf}(\alpha)$, then there exists $1 \leqslant q < n(n-1)/2$ such that each element of $[\alpha^q]^U$ has a standard reduction system.
- If $t_{\sup}(\alpha_{\text{ext}}) < t_{\sup}(\alpha)$, then there exists $1 \leqslant q < n(n-1)/2$ such that each element of $[\alpha^q]^U_{\mathbf{d}}$ has a standard reduction system.

A remark

Commuting elements are useful to the reducibility problem.

Observation. Let $\alpha\beta = \beta\alpha$.

- If β is pseudo-Anosov, α is pseudo-Anosov or periodic.
- If β is reducible and $\mathcal{C}$ is a component of $\mathcal{R}(\alpha)$, then $\mathcal{C}$ is a reduction system of α.
- If β is periodic, we can use projection/lifting method.



Question. How can we find a commuting element, not using iterated cycling on a ultra summit element?